Eur. Phys. J. B **18**, 85–93 (2000)

THE EUROPEAN PHYSICAL JOURNAL B
EDP Sciences

Electromagnetic fields produced by sources on a spherical wavefront

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Received 28 July 1999

Abstract. Transient solutions of the inhomogeneous Maxwell equations are obtained. The specific current source belongs to the spherical surface expanding with the velocity of light. Representations of the electromagnetic field components in the spherical coordinates in terms of the Fourier series are found. The obtained solutions are correct near the wavefront. The general expressions are applied to a description of electromagnetic waves produced by different sources and to a discussion of formation of directional waves and the family of relatively undistorted waves.

PACS. 41.20.Jb Electromagnetic wave propagation; radiowave propagation

1 Introduction

The first consistent consideration of the electromagnetic waves produced by sources moving with velocities both less and greater than the velocity of light was carried out by Heaviside [1]. A general mathematical analysis of the above problem was given by Bateman [2]. Later on the different partial solutions of Maxwell's equations describing the waves produced by the moving current source were considered. We note the explicit solutions, obtained in [3, 4] (pulses of line current moving with the velocity of light), as most connected with the problem discussed in this paper.

We construct the explicit solutions of the initial value problem to the inhomogeneous Maxwell's equations in the space–time domain for the specific sources distributed on the spherical surface expanding with the velocity of light. It should be noted that such sources may be realized, in principle, by means of the pulse of the divergent hard radiation travelling through a medium.

First, we give the algorithm for the construction of the problem in the general case of the radial current. We express the components of the electromagnetic field vectors in the spherical coordinate system and use the variables of the above coordinates. Representing the electromagnetic field vectors in terms of one scalar function, we arrive at the scalar problem. We reduce the obtained scalar equation to the wave equation, which is solved in the variables of the cylindrical coordinates by the Smirnov method of incomplete separation of variables [5] and the Riemann formula. Expressing the solution of the wave equation in terms of modes of the cylindrical coordinate system, we finally represent the components of the electromagnetic field vectors in the spherical coordinate system in terms of the Fourier series only, where the coefficients of the expansions are expressed by means of the inverse Fourier–Bessel transform integral. For the case of the current source distributed on a circle that belongs to the expanding sphere, we find the explicit expressions for the coefficients of the Fourier series in the space–time domain, using the integral relations containing three Bessel functions introduced by Makdonald [6]. It is essential that the obtained solutions are correct near the wavefront.

We apply the obtained expressions for the description of the electromagnetic waves generated by the different point sources moving on the above expanding circle. Each point of the circle moves along a straight line with the velocity of light, therefore, this circle is one among curve types distinguished in [2]. We discuss formation of the family of relatively undistorted waves and construct the directional solutions of the Maxwell equation, including the localized solutions of Brittingham's type [9].

Note that the found representation of the electromagnetic field vectors, produced by radial current, differs from the traditional expansions in terms of the spherical harmonics [7,8].

2 Representation of the transient electromagnetic field produced by a radial current

We construct the solution of the inhomogeneous Maxwell equations for free space

$$\text{curl }\mathbf{D} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial \tau},\ \text{curl }\mathbf{H} = c\frac{\partial \mathbf{D}}{\partial \tau} + \mathbf{j},$$
$$\text{div }\mathbf{D} = q,\ \text{div }\mathbf{H} = 0. \tag{1}$$

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Here $\tau = ct$ is the time variable, c is the velocity of light, $\mathbf{D}$ and $\mathbf{H}$ are the electric induction and the magnetic field strength vectors. The current density vector $\mathbf{j}$ and the charge density q equal zero for $\tau < 0$.

The initial conditions are

$$\mathbf{D} = \mathbf{H} = 0, \quad \tau < 0. \tag{2}$$

We assume that in the spherical coordinates r, ϑ, φ the current density vector has the radial component only

$$\mathbf{j} = j_r \mathbf{e}_r, \tag{3}$$

where $\mathbf{e}_r$ is the unit radial vector. Then, expressing the components of the electric and magnetic field vectors in terms of one scalar function [10]

$$D_r = \frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial \tau^2}, \quad D_\vartheta = \frac{1}{r}\frac{\partial^2 u}{\partial\vartheta\partial r}, \quad D_\varphi = \frac{1}{r\sin\vartheta}\frac{\partial^2 u}{\partial\varphi\partial r},$$
$$H_r = 0, \quad H_\vartheta = \frac{c}{r\sin\vartheta}\frac{\partial^2 u}{\partial\varphi\partial\tau}, \quad H_\varphi = -\frac{c}{r}\frac{\partial^2 u}{\partial\vartheta\partial\tau}, \tag{4}$$

we reduce the vector problem (1) and (2) to the scalar one

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial r^2} - \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) - \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right)\frac{\partial u}{\partial\tau} = \frac{1}{c}j_r,$$
$$\frac{\partial u}{\partial\tau} = 0, \qquad \tau < 0. \tag{5}$$

Hence, using the relation

$$\frac{\partial u}{\partial\tau} = r\psi, \tag{6}$$

we obtain the wave equation and, finally, arrive at the problem

$$\left(\frac{\partial^2}{\partial\tau^2} - \nabla^2\right)\psi = \frac{1}{c}g, \quad \psi \equiv 0, \qquad \tau < 0, \tag{7}$$

where $g = \frac{j_r}{r}$.

We construct the solution of the above problem in variables of the cylindrical coordinate system ρ, φ, z by representing the wavefunction ψ and the component of the current density j_r in the form

$$\psi = \sum_{m=0}^{\infty}\psi_m\begin{pmatrix}\cos m\varphi\\ \sin m\varphi\end{pmatrix}, \quad j_r = \sum_{m=0}^{\infty} j_m\begin{pmatrix}\cos m\varphi\\ \sin m\varphi\end{pmatrix}, \tag{8}$$

where

$$\psi_m = \begin{pmatrix}\psi_m^{\cos}\\ \psi_m^{\sin}\end{pmatrix}, \quad j_m = \begin{pmatrix}j_m^{\cos}\\ j_m^{\sin}\end{pmatrix}.$$

Using the Fourier–Bessel transform

$$F(s,z,\tau) = \int_0^\infty \mathrm{d}\rho\,\rho J_m(s\rho)F(\rho,z,\tau),$$
$$F(\rho,z,\tau) = \int_0^\infty \mathrm{d}s\, sJ_m(s\rho)F(s,z,\tau), \tag{9}$$

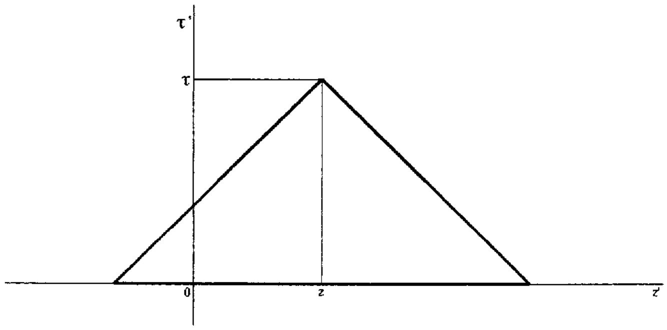

Fig. 1. The triangular integration domain on the z', τ' plane.

where $J_m(s\rho)$ is the Bessel function of the first kind and $F(\rho,z,\tau)$ are the expansion coefficients $\psi_m(\rho,z,\tau)$ or the functions $g_m(\rho,z,\tau) = \frac{1}{\sqrt{\rho^2+z^2}}j_m(\rho,z,\tau)$, we obtain the problem for the functions $\psi_m(s,z,\tau)$

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial\tau^2} - s^2\right)\psi_m(s,z,\tau) = -\frac{1}{c}g_m(s,z,\tau),$$
$$\psi_m(s,z,\tau) \equiv 0, \qquad \tau < 0. \tag{10}$$

The solution of the above problem can be obtained with the help of the Riemann formula

$$\psi_m(s,z,\tau) = \frac{1}{2c}\int_0^\tau \mathrm{d}\tau'\int_{\tau'+z-\tau}^{-\tau'+z+\tau}\mathrm{d}z' g_m(s,z',\tau') \times J_0\left(s\sqrt{(\tau-\tau')^2-(z-z')^2}\right),$$

where $J_0\left(s\sqrt{(\tau-\tau')^2-(z-z')^2}\right)$ is the Riemann function for the telegraphy equation. The triangular integration domain on the plane z', τ' is confined by the lines $\tau' - z' = \tau - z$, $\tau' + z' = \tau + z$ and the axis $\tau' = 0$ (see Fig. 1). Making the inverse Fourier–Bessel transform (9), we get the representations for the coefficients ψ_m in the form

$$\psi_m(\rho,z,\tau) = \frac{1}{2c}\int_0^\tau \mathrm{d}\tau'\int_{\tau'+z-\tau}^{-\tau'+z+\tau}\mathrm{d}z'\int_0^\infty \mathrm{d}s\, sJ_m(s\rho) \times J_0\left(s\sqrt{(\tau-\tau')^2-(z-z')^2}\right)g_m(s,z',\tau'). \tag{11}$$

Collecting the above relation and the expansion (8), we express the scalar wavefunction ψ in terms of modes of the cylindrical coordinate system

$$\psi(\rho,z,\varphi,\tau) = \frac{1}{2c}\sum_{m=0}^{\infty}\begin{pmatrix}\cos m\varphi\\ \sin m\varphi\end{pmatrix} \times \int_0^\tau \mathrm{d}\tau'\int_{\tau'+z-\tau}^{-\tau'+z+\tau}\mathrm{d}z'\int_0^\infty \mathrm{d}s\, sJ_m(s\rho) \times J_0\left(s\sqrt{(\tau-\tau')^2-(z-z')^2}\right)g_m(s,z',\tau'). \tag{12}$$

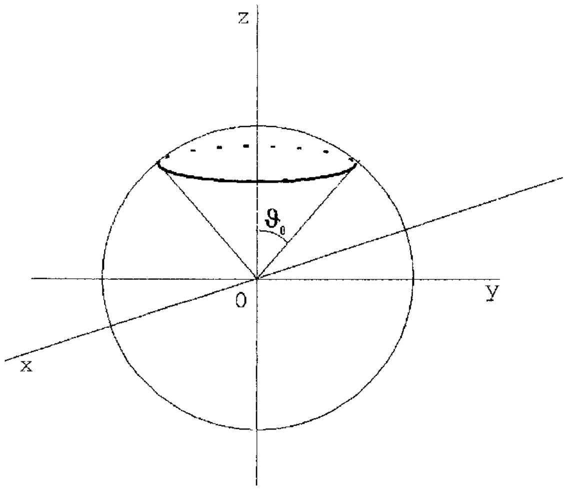

Fig. 2. The formation of the expanding circle. The circumferences 1, 2 have the radiuses $\beta_\perp r_0$ and $\beta_\perp(\tau + r_0)$.

Note that the obtained expression can be interpreted as a wavefunction expansion in terms of the Fourier series. Having obtained the above results and expressions (4, 6), one can obtain the expansions of the field components, differing from traditional representation in terms of spherical harmonics [7].

3 A source on the expanding circle

In this section we give the explicit solution of the scalar problem (7) in the space-time domain for the source current distributed on the specific circle. We assume that the circle is formed as an intersection of the conical surface $\vartheta = \vartheta_0$ with the sphere expanding with the velocity of light (see Fig. 2). The origin of the coordinate system does not coincide with the starting point of the current pulse and the sphere has the radius $r = r_0$ in the initial moment of time. Then we write the component of current density vector in the cylindrical coordinates as

$$j_r = \frac{1}{2\pi}\frac{\delta(\rho - \beta_\perp(\tau + r_0))}{\rho} \times \delta(z - \beta_\parallel(\tau + r_0))f(\varphi, z, \tau), \qquad \tau > 0. \quad (13)$$

Here $f(\varphi, z, \tau)$ is a continuous function, $c\beta_\parallel = c\cos\vartheta_0$ and $c\beta_\perp = c\sin\vartheta_0$ are the projections of the sphere expansion velocity on the axis z, and on the plane $z = \text{const}$, $\beta_\perp^2 + \beta_\parallel^2 = 1$. Hence the circle has the radius $\beta_\perp r_0$ and lies on the plane $z = \beta_\parallel r_0$ $(r_0 > 0)$ in the initial moment of time. Essentially that expression (13) may be incorrect in the initial moment of time when $r_0 = 0$ (see Sect. 4.1).

Using expression (13), we write the expansion coefficients of the function $g = \frac{1}{r}j$

$$g_m(\rho, z, \tau) = \frac{1}{2\pi\sqrt{\rho^2 + z^2}}\frac{\delta(\rho - \beta_\perp(\tau + r_0))}{\rho} \times \delta(z - \beta_\parallel(\tau + r_0))f_m(z, \tau), \qquad \tau > 0,$$

$$g_m(s, z, \tau) = \frac{1}{2\pi\sqrt{\beta_\perp^2(\tau + r_0)^2 + z^2}}J_m(s\beta_\perp(\tau + r_0)) \times \delta(z - \beta_\parallel(\tau + r_0))f_m(z, \tau), \quad (14)$$

and write the solution of the problem (10) in the form

$$\psi_m(s, z, \tau) = \frac{1}{4\pi c}\int_0^\tau d\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} dz' \frac{1}{\sqrt{\beta_\perp^2(\tau' + r_0)^2 + z'^2}} \times \delta(z' - \beta_\parallel(\tau' + r_0))J_m(s\beta_\perp(\tau' + r_0)) \times J_0\left(s\sqrt{(\tau - \tau')^2 - (z - z')^2}\right) f_m(z', \tau').$$

The inverse Fourier–Bessel transform gives

$$\psi_m(\rho, z, \tau) = \frac{1}{4\pi c}\int_0^\tau d\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} dz' \frac{1}{\sqrt{\beta_\perp^2(\tau' + r_0)^2 + z'^2}} \times \delta(z' - \beta_\parallel(\tau' + r_0))f_m(z', \tau') \times \int_0^\infty ds\, sJ_m(s\beta_\perp(\tau' + r_0))J_m(s\rho) \times J_0\left(s\sqrt{(\tau - \tau')^2 - (z - z')^2}\right). \quad (15)$$

Assuming that $\beta_\perp \neq 0$ one can see that the internal integral containing three Bessel functions

$$I_m^{\text{int}}(\rho, z', \tau') = \int_0^\infty ds\, sJ_m(s\beta_\perp(\tau' + r_0))J_m(s\rho) \times J_0\left(s\sqrt{(\tau - \tau')^2 - (z - z')^2}\right)$$

is not equal to zero in the domain on the plane τ', z' determined by the inequalities

$$(\rho - \beta_\perp(\tau' + r_0))^2 < (\tau - \tau')^2 - (z - z')^2 < (\rho + \beta_\perp(\tau' + r_0))^2$$

and is expressed *via* the Legendre function of the first kind $P_{m-1/2}^{1/2}(\cos\theta')$ ([11], expression 6.578)

$$I_m^{\text{int}} = \frac{1}{\rho\beta_\perp(\tau' + r_0)}P_{m-1/2}^{1/2}(\cos\theta')\frac{1}{\sqrt{2\pi\sin\theta'}},$$

where

$$\cos\theta' = \frac{1}{2\beta_\perp\rho(\tau' + r_0)} \times \left(\rho^2 + \beta_\perp(\tau + r_0)^2 + (z - z')^2 - (\tau - \tau')^2\right).$$

Remembering that

$$P_{m-1/2}^{1/2}(\cos\theta') = \sqrt{\frac{2}{\pi\sin\theta'}}\cos m\theta',$$

$$\psi_m(\rho,z,\tau)=\frac{1}{2\pi^2 c((\tau+r_0)^2-z^2-\rho^2)}\int_T^\pi \mathrm{d}\theta\cos m\theta \times f_m\left(\frac{\beta_\parallel((\tau+r_0)^2-z^2-\rho^2)}{2(\tau+r_0-\beta_\parallel z-\beta_\perp\rho\cos\theta)},\frac{(\tau+r_0)^2-z^2-\rho^2}{2(\tau+r_0-\beta_\parallel z-\beta_\perp\rho\cos\theta)}-r_0\right), \quad (19)$$

$$\psi(\rho,\varphi,z,\tau)=\frac{1}{2\pi^2 c((\tau+r_0)^2-z^2-\rho^2)}\sum_{m=0}^{\infty}\begin{pmatrix}\cos m\varphi\\ \sin m\varphi\end{pmatrix}\int_T^\pi \mathrm{d}\theta\cos m\theta \times f_m\left(\frac{\beta_\parallel((\tau+r_0)^2-z^2-\rho^2)}{2(\tau+r_0-\beta_\parallel z-\beta_\perp\rho\cos\theta)},\frac{(\tau+r_0)^2-z^2-\rho^2}{2(\tau+r_0-\beta_\parallel z-\beta_\perp\rho\cos\theta)}-r_0\right), \quad (21)$$

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we write the integral I_m^{int} as

$$I_m^{\text{int}}=\frac{1}{\rho\beta_\perp(\tau'+r_0)}\frac{\cos\, m\theta'}{\pi\sin\theta'}\,. \quad (16)$$

When $\beta_\parallel\neq 0$, $\beta_\perp\neq 0$, the integration domain is the part of the triangle, which is confined by the axis $z'=0$ and the branches of hyperbolas (see Fig. 3)

$$\tau'=\frac{\tau+r_0+\beta_\perp\rho}{\beta_\parallel^2}-\left[\left(\frac{z-z'}{\beta_\parallel}\right)^2+\left(\frac{\rho+\beta_\perp(\tau+r_0)}{\beta_\parallel^2}\right)^2\right]^{1/2},$$
$$\tau'=\frac{\tau+r_0-\beta_\perp\rho}{\beta_\parallel^2}-\left[\left(\frac{z-z'}{\beta_\parallel}\right)^2+\left(\frac{\rho-\beta_\perp(\tau+r_0)}{\beta_\parallel^2}\right)^2\right]^{1/2}. \quad (17)$$

Note that the intersection points of these hyperbolas lie on the line $\tau'=-r_0$.

We calculate the integral (15) by using the property of a δ-function. We find the lower and upper limits of integration with respect to τ' as the coordinates of the intersection points of hyperbolas (17) (or one hyperbola and the axis $z'=0$) with the line $z'-\beta_\parallel(\tau'+r_0)=0$: $T_1=\max\left[0,\frac{\tau^2-z^2-\rho^2-r_0^2+2(\beta_\parallel z-\beta_\perp\rho)r_0}{2(\tau+r_0+\beta_\perp\rho-\beta_\parallel z)}\right]$ and $T_2=\frac{\tau^2-z^2-\rho^2-r_0^2+2(\beta_\parallel z+\beta_\perp\rho)r_0}{2(\tau+r_0-\beta_\perp\rho-\beta_\parallel z)}$, accordingly (see Fig. 3).

Then the expression (15) is reduced to

$$\psi_m(\rho,z,\tau)=\frac{1}{4\pi^2 c\rho\sqrt{1-\beta_\parallel^2}}\int_{T_1}^{T_2}\mathrm{d}\tau'\frac{1}{(\tau'+r_0)^2}\frac{\cos\, m\theta(\tau')}{\sin\theta(\tau')} \times f_m(\beta_\parallel(\tau'+r_0),\tau'), \quad (18)$$

where the angle θ is defined by

$$\cos\theta=\frac{1}{2\beta_\perp\rho(\tau'+r_0)} \times\left(\rho^2+z^2-(\tau+r_0)^2+2(\tau-\beta_\parallel z+r_0)(\tau'+r_0)\right).$$

We can write the integral (18) by changing the variable of integration to

$$\theta=\arccos\left(\frac{\rho^2+z^2-(\tau+r_0)^2}{2\rho\beta_\perp(\tau'+r_0)}+\frac{\tau-\beta_\parallel z+r_0}{\beta_\perp\rho}\right),$$

then

see equation (19) above

where

$$T=\max\left(0,\arccos\frac{1}{2\beta_\perp\rho r_0}(\rho^2+z^2+r_0^2-\tau^2-2\beta_\parallel z r_0)\right). \quad (20)$$

Hence we get the expansion (8) in the form

see equation (21) above

that gives the explicit representation of a wavefunction in terms of the Fourier series only.

In the case $\beta_\perp=1$, $\beta_\parallel=0$, the expanding circle belongs to the immobile plane $z=0$ and expression (20) yields

$$\psi_m(\rho,z,\tau)=\frac{1}{2\pi^2 c((\tau+r_0)^2-\rho^2-z^2)} \times\int_{T(\beta\|=0)}^{\pi}\mathrm{d}\theta\cos m\theta f_m\left(0\,,\frac{(\tau+r_0)^2-z^2-\rho^2}{2(\tau+r_0-\rho\cos\theta)}-r_0\right). \quad (22)$$

Formula (22) and expansion (8) give the representation of the wavefunction ψ produced by a source distributed on the circle expanding with the velocity of light on the immobile plane. Note that result (22) can be obtained in another way by putting $\beta_\perp=1$, $\beta_\parallel=0$ in expression (13). Then the integration domain in (15) is determined by two parabolas

$$\tau^2-(\rho\mp r_0)^2-(z-z')^2=2(\tau+r_0\mp\rho)\tau'.$$

$$\psi_m = \frac{\epsilon_m}{2\pi^2 c((\tau+r_0)^2 - z^2 - \rho^2)} \int\limits_T^{\pi} \mathrm{d}\theta \cos m\theta \begin{pmatrix} \cos m\phi\left(\frac{(\tau+r_0)^2-z^2-\rho^2}{2(\tau+r_0-\beta_\| z-\beta_\perp \rho\cos\theta)} - r_0\right) \\ \sin m\phi\left(\frac{(\tau+r_0)^2-z^2-\rho^2}{2(\tau+r_0-\beta_\| z-\beta_\perp \rho\cos\theta)} - r_0\right)\end{pmatrix}$$
$$\times f\left(\frac{\beta_\|((\tau+r_0)^2-z^2-\rho^2)}{2(\tau+r_0-\beta_\| z-\beta_\perp\rho\cos\theta)}, \frac{(\tau+r_0)^2-z^2-\rho^2}{2(\tau+r_0-\beta_\| z-\beta_\perp\rho\cos\theta)} - r_0\right),$$

$$\psi = \frac{1}{2\pi^2 c((\tau+r_0)^2-z^2-\rho^2)} \sum_{m=0}^{\infty} \epsilon_m \int\limits_T^{\pi} \mathrm{d}\theta \cos m\left(\varphi - \phi\left(\frac{(\tau+r_0)^2-z^2-\rho^2}{2(\tau+r_0-\beta_\| z-\beta_\perp\rho\cos\theta)} - r_0\right)\right)$$
$$\times \cos m\theta f\left(\frac{\beta_\|((\tau+r_0)^2-z^2-\rho^2)}{2(\tau+r_0-\beta_\| z-\beta_\perp\rho\cos\theta)}, \frac{(\tau+r_0)^2-z^2-\rho^2}{2(\tau+r_0-\beta_\| z-\beta_\perp\rho\cos\theta)} - r_0\right).$$

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where $\tau \mp \rho > 0$.

The particular case $\beta_\perp = 0$, $\beta_\| = 1$ is considered in Section 4.1.

The expression (21) permits us, in principle, to obtain the correct solutions of the inhomogeneous Maxwell equations, including the wavefront, for the source on the expanding circle. One can easily get the components of the magnetic field strength vectors H_φ, H_ϑ with the help of (4) and (6) by differentiating solution (21) with respect to the angular variables ϑ, φ. Note that the calculation of the electric field components involves integration with respect to the time variable, which is not, in general, a trivial problem.

4 Application

Let us consider the point source moving on the expanding circle which belongs to the sphere $r = \tau + r_0$. This particular case allows us to discuss the peculiarities of the description of the structure of the wavefunction and electromagnetic field components as near the wavefront as in the domain $\tau - r > r_0$. We get the condition for the simplification of the problem (7), (13) ($r_0 = 0$). We discuss formation of a family of relatively undistorted waves by the point source and directional waves by sources distributed on the above circle.

4.1 Wavefunction representation for a moving point source

We assume that the point source moves with a varying angular velocity on the circle belonging to the expanding sphere and write the relation for the current density in the form

$$j_r = \frac{\delta(\rho - \beta_\perp(\tau+r_0))}{\rho} \times \delta(z - \beta_\|(\tau+r_0))\delta(\varphi - \phi(\tau)) f(z,\tau). \quad (23)$$

One can see that this description of source is incorrect in the initial moment of time if $r_0 = 0$ ($\rho_0 = z_0 = 0$). Note that the circle plane moves with the velocity less than that of light, but the velocity of the point source is equal or greater than the velocity of light.

The coefficients $g_m(\rho,z,\tau) = \frac{1}{\sqrt{\rho^2+z^2}} j_m(\rho,z,\tau)$ are given by

$$g_m = \frac{\epsilon_m}{\pi\rho\sqrt{\rho^2+z^2}} \delta(\rho - \beta_\perp(\tau+r_0))\delta(z - \beta_\|(\tau+r_0)) \times f(z,\tau)\begin{pmatrix}\cos m\phi(\tau) \\ \sin m\phi(\tau)\end{pmatrix},$$

where

$$\epsilon_m = \begin{cases} 1/2, & m = 0 \\ 1, & m \neq 0 \end{cases}.$$

Using the above expression and formula (19) we get

see the first equation above

and write the wavefunction ψ as

see the second equation above

$$\psi = \frac{1}{2\pi c((\tau + r_0)^2 - z^2 - \rho^2)} \int_T^\pi \mathrm{d}\theta\delta \left(\varphi - \theta - \phi\left(\frac{(\tau + r_0)^2 - z^2 - \rho^2}{2(\tau + r_0 - \beta_\| z - \beta_\perp \rho \cos\theta)} - r_0\right)\right) \times f\left(\frac{\beta_\|((\tau + r_0)^2 - z^2 - \rho^2)}{2(\tau + r_0 - \beta_\| z - \beta_\perp \rho \cos\theta)}, \frac{(\tau + r_0)^2 - z^2 - \rho^2}{2(\tau + r_0 - \beta_\| z - \beta_\perp \rho \cos\theta)} - r_0\right). \quad (24)$$

$$\psi = \frac{1}{2\pi c\left((\tau + r_0)^2 - z^2 - \rho^2\right)} h\left(\tau^2 - z^2 - \rho^2 - r_0^2 + 2r_0(\beta_\| z + \beta_\perp \rho \cos(\varphi - \varphi_0))\right) \times f\left(\frac{\beta_\|((\tau + r_0)^2 - z^2 - \rho^2)}{2(\tau + r_0 - \beta_\| z - \beta_\perp \rho \cos(\varphi - \varphi_0))}, \frac{(\tau + r_0)^2 - z^2 - \rho^2}{2(\tau + r_0 - \beta_\| z - \beta_\perp \rho \cos(\varphi - \varphi_0))} - r_0\right), \quad (25)$$

Interchanging the summation and integration in the above expression and using the relation $\delta(\varphi - \phi(\theta) - \theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{m=1}^{\infty} \cos m(\varphi - \phi(\theta)) \cos m\theta$, we arrive at the following result

see equation (24) above.

Hence to obtain the explicit expression for the wavefunction, we have to solve the equation $\varphi - \theta - \phi\left(\frac{(\tau+r_0)^2-z^2-\rho^2}{2(\tau+r_0-\beta_\| z-\beta_\perp \rho\cos\theta)} - r_0\right) = 0$ with respect to the variable of integration θ. Using the obtained expressions, one can get the correct representations of the wavefunction in the case of the point source moving on the expanding circle.

4.2 Electromagnetic field representation for the point source

Using the results of Section 4.1, we obtain the expressions for the wavefunction and the components of the magnetic strength vector for the source moving along the straight line with the velocity of light and get the condition permitting simplification of problem (7, 13).

Let the point source move along the straight line deflected from the axis z by the angle ϑ_0. Then the function $\phi(\tau) = \varphi_0$ does not depend on the time variable and the integration in expression (24) can be easily performed. Choosing the integration limit in (20) $T = \arccos \frac{1}{2\beta_\perp \rho r_0}\left(\rho^2 + z^2 + r_0^2 - \tau^2 - 2\beta_\| z r_0\right)$ we obtain the expression correct near the wavefront

see equation (25) above

where

$$h(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

is the Heaviside function defined by the argument of the δ-function and the limit of integration T.

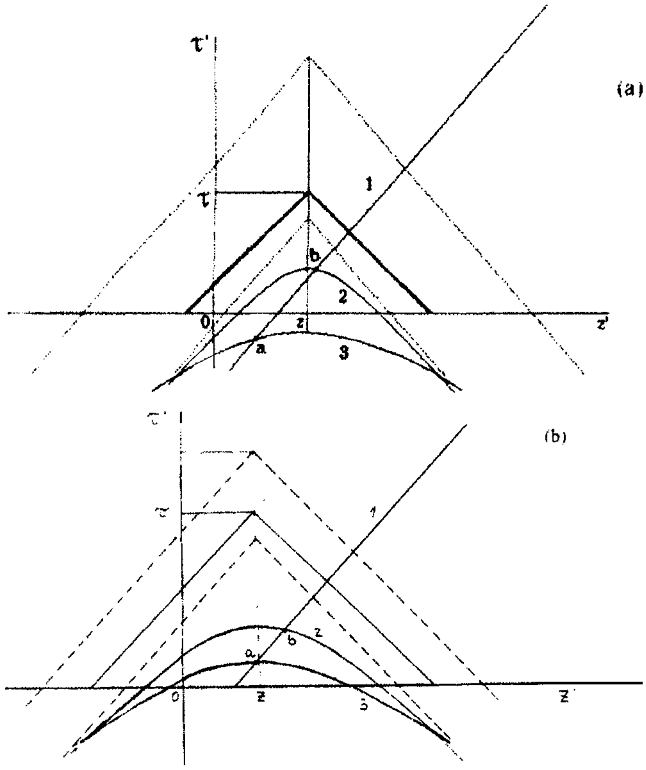

Fig. 3. The usage of the z', τ' plane diagrams for the definition of limits of integration in expression (18); 1 – the line $z' - \beta_\| \tau - \beta_\| r_0 = 0$; 2, 3 – the hyperbolas (16); "- - -" – the asymptotes of the hyperbolas; a, b – the intersection points of the hyperbolas 3 and 2 with the line 1; (a) the case $\tau^2 - z^2 - \rho^2 - r_0^2 + 2(\beta_\| z - \beta_\perp \rho) r_0 > 0$, $T_1 = 0$; (b) the case $\tau^2 - z^2 - \rho^2 - r_0^2 + 2(\beta_\| z - \beta_\perp \rho) r_0 < 0$; $T_1 = \tau_a$, τ_a is the τ coordinate of the point a.

Writing the above expression in the variables of the spherical coordinate system

$$\psi = \frac{1}{2\pi c\left((\tau + r_0)^2 - r^2\right)} h\left(\tau^2 - r^2 - r_0^2 + 2rr_0 \cos\Theta\right) \times f\left(\frac{\beta_\|((\tau + r_0)^2 - r^2)}{2(\tau + r_0 - r\cos\Theta)}, \frac{(\tau + r_0)^2 - r^2}{2(\tau + r_0 - r\cos\Theta)} - r_0\right), \quad (26)$$

where

$$\cos\Theta = \cos\vartheta_0 \cos\vartheta + \sin\vartheta_0 \sin\vartheta \cos(\varphi - \varphi_0),$$

we get the components of the magnetic field strength vector for the source moving along the straight line from the origin of the coordinate system by using the relations (4, 6) and taking the limit $r_0 \to 0$

$$H_\varphi = H_{\varphi\delta} + H_{\varphi h} = \frac{\beta_\| \sin\vartheta - \beta_\perp \cos\vartheta \cos(\varphi - \varphi_0)}{4\pi r(1 - \cos\Theta)} \times \delta(\tau - r) f(0,0) - \frac{1}{2\pi(\tau^2 - r^2)} h(\tau - r) \times \frac{\partial}{\partial\vartheta} f\left(\frac{\beta_\|(\tau^2 - r^2)}{2(\tau - r\cos\Theta)}, \frac{\tau^2 - r^2}{2(\tau - r\cos\Theta)}\right), \quad (27)$$

$$H_\vartheta = H_{\vartheta\delta} + H_{\vartheta h} = \frac{1}{4\pi}\delta(\tau - r)\frac{\beta_\perp \sin(\varphi - \varphi_0)}{r(1 - \cos\Theta)} f(0,0) + \frac{1}{2\pi} h(\tau - r) \times \frac{1}{\sin\vartheta(\tau^2 - r^2)} \frac{\partial}{\partial\varphi} f\left(\frac{\beta_\|(\tau^2 - r^2)}{2(\tau - r\cos\Theta)}, \frac{\tau^2 - r^2}{2(\tau - r\cos\Theta)}\right). \quad (28)$$

We assume that the point source moves from the origin of the coordinate system ($r_0 = 0$) and write the source (13) as

$$j_r(\rho, \varphi, z, \tau) = \frac{\delta(\rho - \beta_\perp \tau)}{\rho} \delta(z - \beta_\| \tau) \delta(\varphi - \varphi_0) f(z, \tau).$$

Then we get from (24), where $r_0 = 0$, $\phi = \varphi_0$,

$$\psi = \frac{1}{2\pi c(\tau^2 - r^2)} f\left(\frac{\beta_\|(\tau^2 - r^2)}{2(\tau - r\cos\Theta)}, \frac{\tau^2 - r^2}{2(\tau - r\cos\Theta)}\right). \quad (29)$$

Using formula (4) and expression (29), we describe the magnetic field vector components by the second terms $H_{\varphi h}$, $H_{\vartheta h}$ of expressions (27, 28). These results are correct in the space-time domain $\tau - r > 0$ and on the wavefront if the condition $f|_{\tau=0} = 0$ is true. It should be noted that this condition is also necessary in the common case of the angular source representation $\delta(\varphi - \phi(\tau))$.

4.3 On the family of relatively undistorted waves

Let us write expression (29) in the form

$$\psi = \frac{1}{2\pi c\,(\tau^2 - z^2 - \rho^2)} f\left(\frac{\tau^2 - z^2 - \rho^2}{2(\tau - \beta_\| z - \beta_\perp \rho\cos(\varphi - \varphi_0))}\right). \quad (30)$$

One can see that the above expression describes the family of relatively undistorted progressing waves with distortion factor $\frac{1}{\tau^2 - z^2 - \rho^2}$ and the phase function $\frac{\tau^2 - z^2 - \rho^2}{2(\tau - \beta_\| z - \beta_\perp \rho\cos(\varphi - \varphi_0))}$, here we use Courant, Hilbert terms [12].

When $\beta_\perp = \sin\vartheta_0 = 0, \beta_\| = \cos\vartheta_0 = 1$, one gets from (30) the wavefunction

$$\psi = \frac{1}{2\pi c\,(\tau^2 - z^2 - \rho^2)} f\left(\frac{1}{2}\left(\tau + z - \frac{\rho^2}{\tau - z}\right)\right), \quad (31)$$

akin to the relatively undistorted axial symmetric solution of a homogeneous wave equation $\psi = \frac{1}{\tau - z} f\left(\tau + z - \frac{\rho^2}{\tau - z}\right)$ [13], describing the family of localized waves of Bessel-Gauss type, and different from it by the distortion factor only.

Note that expression (31) can be obtained with the help of Bateman's transfer [13,14]

$$f'(\rho, z, \tau) \to \psi(\rho, z, \tau) \equiv \frac{1}{z - \tau} \times f'\left(\frac{\rho}{z - \tau}, \frac{\rho^2 + z^2 - \tau^2 - 1}{2(z - \tau)}, \frac{\rho^2 + z^2 - \tau^2 + 1}{2(z - \tau)}\right)$$

from the function $f' = -\frac{1}{\tau + z} f\left(-\frac{2\alpha}{\tau + z}\right)$.

If $\beta_\perp = \sin\vartheta_0 = 1, \beta_\| = \cos\vartheta_0 = 0$, then (30) gives

$$\psi = \frac{1}{2\pi c\,(\tau^2 - \rho^2 - z^2)} f\left(\frac{\tau^2 - z^2 - \rho^2}{2(\tau - \rho\cos(\varphi - \varphi_0))}\right). \quad (32)$$

In order to get the above result the arbitrary function f in expression (23) has to necessarily depend on the time variable τ.

Expression (30) and results of Section 4.2 allow us to discuss the opportunity of formation of relatively undistorted waves by different point sources. It should be noted that results (27) and (28) can be interpreted as relatively undistorted solutions of Maxwell equations in the space-time domain $\tau - r > 0$.

4.4 Formation of directional waves

Let us apply the obtained results of Section 3 to construct the specific directional solutions of the wave equation. We assume that the source is distributed on the circle which begins its expansion from the origin of the coordinate system $r_0 = 0$, and obtain the expression for the mth wave mode $\psi_m(\rho, \varphi, z, \tau) = \psi_m e^{im\varphi}$ in the space-time domain $\tau - r > 0$ only. We write the function $f_m(z, \tau)$ in expression (14) as $\exp(-\frac{\alpha}{\tau})$, where α is a positive constant. Then one gets the coefficients $\psi_m(\rho, z, \tau)$ from (19) in the form

$$\psi_m = \frac{c_m}{2\pi^2 c(\tau^2 - z^2 - \rho^2)} \exp\left(\frac{-2\alpha(\tau - \beta_\| z)}{\tau^2 - z^2 - \rho^2}\right) \times \int_0^\pi d\theta\, \cos m\theta \exp\left\{\frac{2\alpha\beta_\perp \rho}{\tau^2 - z^2 - \rho^2}\cos\theta\right\}.$$

Remembering that $\int_0^\pi \mathrm{d}\theta \ \cos m\theta \mathrm{e}^{z\cos\theta} = \pi I_m(z)$ ([11], 8.431.5), where $I_m(z)$ is the modified Bessel function of the first kind, we obtain the wavefunction of order m

$$\psi_m(\rho,z,\varphi,\tau)=\frac{c_m \mathrm{e}^{\mathrm{i}m\varphi}}{2\pi c(\tau^2-z^2-\rho^2)} \times \exp\left\{-\frac{2\alpha(\tau-\beta_\parallel z)}{\tau^2-z^2-\rho^2}\right\} I_m\left(\frac{2\alpha\beta_\perp\rho}{\tau^2-z^2-\rho^2}\right). \quad (33)$$

which is similar to the focus wave modes of Bessel-Gauss type [15,16]. Note that the latter can not be reduced to (33) by any parallel translation and rotation of the coordinate system.

When the argument $\frac{2\beta_\perp\alpha\rho}{\tau^2-z^2-\rho^2}$ of the modified Bessel function is large, the mth term of the above expansion can be represented as

$$\psi_m(\rho,z,\varphi,\tau)\simeq\frac{c_m \mathrm{e}^{\mathrm{i}m\varphi}}{4\pi c}\left[\frac{1}{\pi\alpha\beta_\perp\rho(\tau^2-z^2-\rho^2)}\right]^{1/2} \times \exp\left\{-\frac{2\alpha}{\tau^2-z^2-\rho^2}(\tau-\beta_\parallel z-\beta_\perp\rho)\right\}. \quad (34)$$

Let us consider expression (33) in the limiting cases.

(i) When the point source moves with the velocity of light along the axis z ($\beta_\perp=0$), the axial symmetric mode differs from zero only

$$\psi(\rho,\varphi,z,\tau)=\psi_0(\rho,z,\tau)=\frac{c_0}{2\pi c(\tau^2-z^2-\rho^2)} \times \exp\left\{-2\alpha\left(\tau+z-\frac{\rho^2}{\tau-z}\right)^{-1}\right\}. \quad (35)$$

The above wavefunction tends to zero as $\tau^2-z^2-\rho^2\to 0$ ($\tau-z\neq 0$) and describes the space-time structure of a wave localized in the domain close to the axis z. Result (35) is the particular case of expression (31).

(ii) When the source is distributed on a circle lying on the immobile plane $z=0$ ($\beta_\parallel=0$) and expanding with the velocity of light we get from (33)

$$\psi_m(\rho,z,\varphi,\tau)=\frac{c_0 \mathrm{e}^{\mathrm{i}m\varphi}}{2\pi c(\tau^2-z^2-\rho^2)} \times \exp\left\{-\frac{2\alpha\tau}{\tau^2-z^2-\rho^2}\right\} I_m\left(\frac{2\alpha\rho}{\tau^2-z^2-\rho^2}\right). \quad (36)$$

For the large value of the argument $\frac{2\alpha\rho}{\tau^2-z^2-\rho^2}$ we arrive at the following expression

$$\psi_m(\rho,z,\tau)\simeq\frac{c_m \mathrm{e}^{\mathrm{i}m\varphi}}{4\pi c}\left[\frac{1}{\pi\alpha\rho(\tau^2-z^2-\rho^2)}\right]^{1/2} \times \exp\left\{-2\alpha\left(\tau+\rho-\frac{z^2}{\tau-\rho}\right)^{-1}\right\}. \quad (37)$$

The above result describes the directional structure of the wave modes $\psi_m(\rho,\varphi,z,\tau)$ in the vicinity of the plane $z=0$.

In the general case the directionality of the formed wave modes is governed by the current density vector.

Note that one can easily get the terms of the expansions of the magnetic field strength components by differentiating the wave functions $\psi_m(\rho,z,\tau)\mathrm{e}^{\mathrm{i}m\varphi}$ with respect to the angular variables φ and ϑ.

5 Conclusion remarks

The completed investigation was stimulated by the problem of the description of the electromagnetic waves produced by a pulse of hard radiation travelling through a medium. Recall the physical basis of the macroscopic electric current formation. When the main processes of interaction of hard radiation quanta with atoms of the medium are photoabsorption and the Compton effect, then the angular distributions of photoelectrons and compton-electrons have a certain directionality towards the direction of propagation of the X-ray pulse. Therefore, absorption of the energy of the pulse of X-rays results in a partially regular motion of the electrons, *i.e.* the formation of the electric current. The space–time structure of the current distribution is determined by parameters and the structure both of the X-ray pulse and the domain absorbing the quanta. Thus, the point source moving with the velocity greater than light (simplified model in Sect. 4) may be formed by the δ-pulse of the divergent hard radiation and the absorbing domain having the shape of the curve on the conical surface.

The expansions in terms of modes of the cylindrical coordinate system allow one to give adequate description of the peculiarities of the space–time structure of the wavefunctions and, consequently, of the electromagnetic fields produced by the divergent hard radiation.

Note that the transient fields have alternative descriptions in terms of spherical harmonics which can be obtained by using the results obtained in [7,8]. This description is beyond the scope of the present paper.

The research described in this publication was made possible in part by Grant 99-02-16893 from the Russian Basic Research Fund.

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